

DUALITY OF DISCRETE TOPOLOGICAL VECTOR SPACES

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Abstract. For a field \mathbb{F} , the discrete topological vector spaces over \mathbb{F} are essentially of the form \mathbb{F}^α where α is an ordinal. With additional appropriate properties, they are isomorphic to $\mathbb{F}^{(\beta)}$ where β is again an ordinal. Finally, the categories of the vector spaces of the first and the second type are equivalent.

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1. Introduction. Let \mathbb{F} be a field and \mathbb{N} the set of the natural integers. We write $\mathbb{F}^{\mathbb{N}}$ for the set of all sequences $y : \mathbb{N} \longrightarrow \mathbb{F}, i \longmapsto y_i$ and $\mathbb{F}^{(\mathbb{N})}$ for the subset of $\mathbb{F}^{\mathbb{N}}$ consisting of the sequences y with finite support. These are vector spaces over \mathbb{F} . The field \mathbb{F} is endowed with the *discrete topology* and becomes a topological vector space. With the product topology, $\mathbb{F}^{\mathbb{N}}$ becomes a topological vector space too, whose 0-basis consists of sets $(V_n)_{n \in \mathbb{N}^*}$ where for all $n \in \mathbb{N}^*$,

$$V_n = \{y \in \mathbb{F}^{\mathbb{N}} \mid y_i = 0 \text{ for all } i < n\}.$$

It follows immediately that

$$V_1 \supset V_2 \supset \dots \supset V_n \supset V_{n+1} \supset \dots$$

Let \mathfrak{G} be the category of the discrete \mathbb{F} -vector spaces : an object of \mathfrak{G} is a vector space with a denumerable algebraic basis. If U and V are two objects of \mathfrak{G} , a morphism $f : U \longrightarrow V$ is just a linear map from U to V . We write $\mathbf{Hom}_{\mathfrak{G}}(U, V)$ for the set of morphisms from U to V .

The category $\mathcal{T}\mathfrak{G}$ is the subcategory of \mathfrak{G} whose objects are the topological (discrete) vector spaces (W, \mathcal{T}_W) verifying the following properties :

- (1) A basis of the filter of neighbourhoods of 0 consists of the lattice \mathcal{C} of the subspaces C of W of finite codimension,
- (2) The topology \mathcal{T}_W is Hausdorff, i.e

$$\bigcap_{C \in \mathcal{C}} C = \{0\},$$

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(3) Le lattice \mathcal{C} has a denumerable subset $\mathcal{B} = \{V_n \mid n \in \beta\}$ (where β is an ordinal) which verifies

$$W = V_0 \supset V_1 \supset \dots \supset V_n \supset \dots \quad \text{with} \quad \dim(W/V_n) = n.$$

(4) The topology \mathcal{T}_W is complete, i.e

$$W = \varprojlim (W/V_n).$$

For every two objects U and V of $\mathcal{T}\mathfrak{G}$, a morphism $f : U \longrightarrow V$ is a continuous linear mapping. We write $\mathbf{Hom}_{\mathcal{T}\mathfrak{G}}(U, V)$ for the set of morphisms from U to V .

The functor $\mathbf{Hom}_{\mathbb{F}}(-, \mathbb{F})$ is defined by

$$\begin{aligned} \mathbf{Hom}_{\mathbb{F}}(-, \mathbb{F}) : \mathfrak{G} &\longrightarrow \mathcal{T}\mathfrak{G} \\ U &\longmapsto \mathbf{Hom}_{\mathbb{F}}(U, \mathbb{F}) \\ \mathbf{Hom}_{\mathfrak{G}}(U, V) &\longmapsto \begin{cases} \mathbf{Hom}_{\mathbb{F}}(f, \mathbb{F}) \in \mathbf{Hom}_{\mathcal{T}\mathfrak{G}}(\mathbf{Hom}_{\mathbb{F}}(V, \mathbb{F}), \mathbf{Hom}_{\mathbb{F}}(U, \mathbb{F})) \\ \mathbf{Hom}_{\mathbb{F}}(f, \mathbb{F})(y) = y \circ f \quad \text{for all } y \in \mathbf{Hom}_{\mathbb{F}}(V, \mathbb{F}). \end{cases} \end{aligned} \quad (1.1)$$

Our goal is to show that this functor induces a duality ([2], [4], [5]) between the categories \mathfrak{G} and $\mathcal{T}\mathfrak{G}$. For this purpose, parting from $\mathbb{F}^{\mathbb{N}}$, we introduce the topological vector spaces \mathbb{F}^{α} and $\mathbb{F}^{(\alpha)}$ where α is an ordinal. Using inverse limits of sets and categories ([1]), we characterise \mathfrak{G} and $\mathcal{T}\mathfrak{G}$ in section 2. The main result is stated in theorem 3.2.

2. The topological vector space of sequences over a field. We begin with a lemma, which characterizes of $\mathbb{F}^{\mathbb{N}}$:

LEMMA 2.1. *The following properties hold :*

- (1) $\mathbb{F}^{\mathbb{N}}/V_n \simeq \mathbb{F}^n$ for all $n \in \mathbb{N}^*$.
- (2) For every $m, n \in \mathbb{N}^*$ such that $m \leq n$, let

$$\begin{aligned} g_{m,n} : \mathbb{F}^{\mathbb{N}}/V_n &\longrightarrow \mathbb{F}^{\mathbb{N}}/V_m \\ \bar{x}^n &\longrightarrow \bar{x}^m \end{aligned} \quad (2.1)$$

where for all $x \in \mathbb{F}^{\mathbb{N}}$, \bar{x}^k is the class of x modulo V_k . Then

$$\varprojlim (\mathbb{F}^{\mathbb{N}}/V_n) \simeq \mathbb{F}^{\mathbb{N}} \quad (2.2)$$

where $\varprojlim (\mathbb{F}^{\mathbb{N}}/V_n)$ is the **inverse limit** of the sequence $(\mathbb{F}^{\mathbb{N}}/V_n)_{n \in \mathbb{N}^*}$ with respect to the $g_{m,n}$

Proof. (1) For all $n \in \mathbb{N}^*$, we have

$$\mathbb{F}^{\mathbb{N}} = V_n \oplus U_n$$

where

$$\begin{aligned} U_n &= \{y \in \mathbb{F}^{\mathbb{N}} \mid y_i = 0 \text{ for all } i \geq n\} \\ &= \{(y_1, y_2, \dots, y_n, 0, \dots, 0, \dots) \mid y_i \in \mathbb{F} \text{ for all } i = 1, \dots, n\}. \end{aligned}$$

Thus $U_n \simeq \mathbb{F}^n$ and $\mathbb{F}^{\mathbb{N}}/V_n \simeq U_n \simeq \mathbb{F}^n$.

(2) Given $m, n \in \mathbb{N}^*$ with $m \leq n$, consider the mapping

$$\begin{aligned} f_{m,n} : \mathbb{F}^n &\longrightarrow \mathbb{F}^m \\ (x_1, \dots, x_m, x_{m+1}, \dots, x_n) &\longmapsto (x_1, x_2, \dots, x_m). \end{aligned}$$

By definition, the *inverse limit* ([1]) of the sequence $(\mathbb{F}^n)_{n \in \mathbb{N}^*}$ with respect to these mapping is

$$\begin{aligned} \varprojlim (\mathbb{F}^n) &= \{(x_1, \dots, x_{n+1}, \dots) \in \prod_{n \in \mathbb{N}^*} \mathbb{F}^n \mid f_{m,n}(x_n) = x_m \ \forall m, n \in \mathbb{N}^*, m \leq n\} \\ &= \{(x_1, (x_1, x_2), \dots, (x_1, \dots, x_{n-1}), (x_1, \dots, x_n), (x_1, \dots, x_{n+1}), \dots)\} \subset \prod_{n \in \mathbb{N}^*} \mathbb{F}^n, \end{aligned} \quad (2.3)$$

so that, on the one hand,

$$\varprojlim (\mathbb{F}^n) \simeq \mathbb{F}^{\mathbb{N}},$$

and on the other hand, by the isomorphism

$$\begin{aligned} \mathbb{F}^{\mathbb{N}}/V_n &\longrightarrow \mathbb{F}^n \\ \bar{x}^n &\longrightarrow (x_1, \dots, x_n), \end{aligned}$$

for any $n \in \mathbb{N}^*$, we get, by taking the inverse limit of the sequence $(\mathbb{F}^{\mathbb{N}}/V_n)_{n \in \mathbb{N}}$ with respect to the mappings $f_{m,n}$:

$$\varprojlim (\mathbb{F}^{\mathbb{N}}/V_n) = \varprojlim (\mathbb{F}^n) \simeq \mathbb{F}^{\mathbb{N}}.$$

□

Now we state the main result of the section

PROPOSITION 2.2. (1) Any object of $\mathcal{T}\mathfrak{G}$ (resp. of \mathfrak{G}) is isomorphic to some \mathbb{F}^α (resp. $\mathbb{F}^{(\alpha)}$) where α is an ordinal.

(2) Any morphism $f : \mathbb{F}^{(\beta)} \longrightarrow \mathbb{F}^{(\alpha)}$ is of the form

$$x \longmapsto x \cdot \mathcal{F}$$

where $\mathcal{F} \in \mathbb{F}^{\beta, (\alpha)}$.

(3) Any morphism $g : \mathbb{F}^\alpha \longrightarrow \mathbb{F}^\beta$ is of the form

$$y \longmapsto \mathcal{G} \cdot y$$

where $\mathcal{G} \in \mathbb{F}^{\beta,(\alpha)}$.

Proof. (1) For the category $\mathcal{T}\mathfrak{G}$: for $W \in \mathcal{T}\mathfrak{G} \setminus \{0\}$, if $\dim W = n < +\infty$, then $W \simeq \mathbb{F}^n$. So, assume that $\dim W = +\infty$; it is known that there are $n \in \mathbb{N}^*$ and a subspace U_n of W such that

$$W = V_n \oplus U_n \text{ with } W/V_n \simeq U_n, \dim U_n = n \text{ and } U_m \subset U_n \text{ for all } m \leq n.$$

Now, consider the category whose objets are respectively the sets $(W/V_n), U_n$ et \mathbb{F}^n endowed, for all $m \leq n$, with the morphisms $f_{m,n}$ and $g_{m,n}$ as in the lemma (2.1) and its proof and the projections $h_{m,n} : U_n \longrightarrow U_m$ of U_n on U_m . By taking the inverse limits with respect to these categories and using the lemma (2.1), we obtain

$$W = \varprojlim (W/V_n) \simeq \varprojlim (U_n) \simeq \mathbb{F}^{\mathbb{N}}.$$

For the category \mathfrak{G} : If $P \in \mathfrak{G}$ and $\dim P = n < +\infty$, then $P \simeq \mathbb{F}^n = \mathbb{F}^{(n)}$. So, assume that $\dim P = +\infty$. Let $(b_n)_{n \in \mathbb{N}}$ a basis of P ; any element $v \in P$ can be uniquely expressed as

$$v = \sum_{n \in \mathbb{N}} \lambda_n b_n$$

where the sequence $(\lambda_n)_{n \in \mathbb{N}}$ is with finite support, i.e an element of $\mathbb{F}^{(\mathbb{N})}$. Thus $P \simeq \mathbb{F}^{(\mathbb{N})}$.

(2) Any morphism $f : \mathbb{F}^{(\beta)} \longrightarrow \mathbb{F}^{(\alpha)}$ is defined by its matrix with respect to the canonical bases $\delta = (\delta_i)_{i \in \beta}$ of $\mathbb{F}^{(\beta)}$ and $\delta' = (\delta'_j)_{j \in \alpha}$ of $\mathbb{F}^{(\alpha)}$. This matrix is of the form

$$\mathcal{F} = (d_{ji})_{j \in \beta, i \in \alpha} \text{ with } d_{ji} \in \mathbb{F} \text{ for all } (j, i) \in \beta \times \alpha.$$

Since for any $j \in \beta$, we have de definite sum

$$\mathcal{F}(\delta_j) = \sum_{i \in \alpha} d_{ji} \delta'_i,$$

the sequence $(\delta_{ji})_{i \in \alpha}$ is necessarily with finite support. Thus $\mathcal{F} \in \mathbb{F}^{\beta \times (\alpha)}$ and conversely, such a matrix defines a morphism $f : \mathbb{F}^{(\beta)} \longrightarrow \mathbb{F}^{(\alpha)}$ by the equation (2.2).

(3) Let \mathcal{G} be defined by the equation (2.2). For any $y \in \mathbb{F}^\alpha$, we have, for all $j \in \beta$

$$(\mathcal{G} \cdot y)_j = \sum_{i \in \alpha} \mathcal{G}_{ji} y_i$$

and since this sum is definite, the sequence $(\mathcal{G}_{ji})_{i \in \alpha}$ is necessarily with finite support, i.e $\mathcal{G} \in \mathbb{F}^{\beta,(\alpha)}$, and such a matrix defines a morphism $g : \mathbb{F}^\alpha \longrightarrow \mathbb{F}^\beta$ by the equation (2.2). This morphism is continuous in 0, therefore continuous on \mathbb{F}^α .

Conversely, let $g : \mathbb{F}^\alpha \longrightarrow \mathbb{F}^\beta$ be a continuous linear mapping. If $\alpha \in \mathbb{N}$, then the existence of g in the equation (2.2) is trivial, so assume that $\alpha = \mathbb{N}$. For any $y \in \mathbb{F}^{\mathbb{N}}$, we have $y = \lim y^{(n)}$ for the topology of $\mathbb{F}^{\mathbb{N}}$ where

$$y^{(n)} = (y_1, y_2, \dots, y_n, \dots, 0, \dots) \in \mathbb{F}^{(\mathbb{N})}.$$

Using the continuity of g , we have

$$g(y) = \lim g(y^{(n)}).$$

By the injections $\mathbb{F}^{(n)} \hookrightarrow \mathbb{F}^{(\mathbb{N})} \hookrightarrow \mathbb{F}^{\mathbb{N}}$, we may see $\mathbb{F}^{(n)}$ as a subspace of $\mathbb{F}^{\mathbb{N}}$. Let $g^{(n)}$ be the restriction of g to $\mathbb{F}^{(n)}$; it is defined by a matrix $\mathcal{G}^{(n)} = (\mathcal{G}_{ji}^{(n)}) \in \mathbb{F}^{\beta, n}$, i.e.

$$g^{(n)}(y^{(n)}) = \mathcal{G}^{(n)} \cdot y^{(n)}.$$

Now, for any $i \in \mathbb{N}$ such that $i \leq n$, we have

$$g(\delta_i) = g^{(n)}(\delta_i) \cdot \mathcal{G}^{(n)}(\delta_i)$$

and for any $j \in \beta$,

$$g(\delta_i)_j = \sum_{\nu \in \mathbb{N}} \mathcal{G}_{j\nu}^{(n)} \cdot (\delta_i)_\nu = \mathcal{G}_{ji}^{(n)}.$$

It follows that, for any $j \in \beta$,

$$g^{(n)}(y_j^{(n)}) = \sum_{i \in \mathbb{N}} \mathcal{G}_{ji}^{(n)} \cdot y_i^{(n)} = \sum_{i=1}^n (g(\delta_i))_j \cdot y_i.$$

Hence the sequence $(\sum_{i=1}^n (g(\delta_i))_j y_i)_{n \in \mathbb{N}}$ converges to $(g(y))_j$ in \mathbb{F} . Since the topology of \mathbb{F} is discrete, this implies that the sequence $(\sum_{i=1}^n (g(\delta_i))_j y_i)_{n \in \mathbb{N}}$ is stationary. Thus the sequence $(g(\delta_i)_j)_{i \in \mathbb{N}}$ is necessarily with finite support. Therefore, there exists $N \in \mathbb{N}$ such that

$$g(y)_j = \lim \sum_{i=1}^n g((\delta_i))_j y_i = \sum_{i=1}^{N_j} g(\delta_i)_j y_i.$$

Now, let $\mathcal{G} \in \mathbb{F}^{\beta, \alpha}$ be the matrix defined by $\mathcal{G}_{ji} = g(\delta_i)_j$. Then $\mathcal{G} \in \mathbb{F}^{\beta, (\alpha)}$ and

$$g(y) = \mathcal{G} \cdot y \quad \text{for all } y \in \mathbb{F}^{\mathbb{N}}.$$

□

3. Duality. In the equation (1.1), we must show that the functor $\mathbf{Hom}_{\mathbb{F}}(f, \mathbb{F})$ is an element of $\mathbf{Hom}_{\mathcal{T}\mathfrak{G}}(\mathbf{Hom}_{\mathbb{F}}(V, \mathbb{F}), \mathbf{Hom}_{\mathbb{F}}(U, \mathbb{F}))$. We may take $U = \mathbb{F}^{\alpha}$ and $V = \mathbb{F}^{\beta}$ with $\alpha, \beta \in \mathbb{N}$. For any $y \in \mathbb{F}^{\beta}$, the matrix of the linear mapping $f : \mathbb{F}^{\beta} \longrightarrow \mathbb{F}$ is $\mathcal{H} \in \mathbb{F}^{\beta, 1}$ with

$$y(\delta_i) = y_i = \delta_i \cdot \mathcal{H} = \sum_{j \in \beta} (\delta_i)_j \cdot \mathcal{H} = \mathcal{H}_{i1}$$

for any $i \in \beta$. If \mathcal{F} is the matrix of $f : \mathbb{F}^{(\beta)} \longrightarrow \mathbb{F}^{(\alpha)}$, then the matrix of the linear mapping $y \circ f$ is $\mathcal{H} \circ \mathcal{F}$. Thus it is a continuous linear mapping with respect to the topology $\mathcal{T}\mathfrak{G}$.

The functor $\mathbf{Hom}_{\mathbb{F}}(-, \mathbb{F})$ is exact, i.e. transforms an exact sequence of vector spaces of \mathfrak{G}

$$0 \longrightarrow U \xrightarrow{f} V$$

to the exact sequence of vector spaces of $\mathcal{T}\mathfrak{G}$

$$0 \longleftarrow \mathbf{Hom}_{\mathbb{F}}(U, \mathbb{F}) \xleftarrow{\mathbf{Hom}_{\mathbb{F}}(f, \mathbb{F})} \mathbf{Hom}_{\mathbb{F}}(V, \mathbb{F}).$$

This is because any subspace of a vector space admits a supplementary vector space and a linear mapping is defined by its restrictions to all its supplementary subspaces.

PROPOSITION 3.1. *Let $\alpha, \beta \in \mathbb{N}$ and a linear mapping*

$$\begin{aligned} f : \mathbb{F}^{(\beta)} &\longrightarrow \mathbb{F}^{(\alpha)} \\ x &\longmapsto x \cdot \mathcal{F} \end{aligned} \tag{3.1}$$

where $\mathcal{F} \in \mathbb{F}^{\beta, 1}$. Then \mathcal{F} is also the matrix of $\mathbf{Hom}_{\mathbb{F}}(f, \mathbb{F})$, i.e.

$$\begin{aligned} \mathbf{Hom}_{\mathbb{F}}(f, \mathbb{F}) : \mathbb{F}^{\alpha} &\longrightarrow \mathbb{F}^{\beta} \\ y &\longmapsto \mathcal{F} \cdot y. \end{aligned}$$

Proof. Applying the functor $\mathbf{Hom}_{\mathbb{F}}(-, \mathbb{F})$ to the equation (3.1), we obtain

$$\begin{aligned} \mathbf{Hom}_{\mathbb{F}}(f, \mathbb{F}) : \mathbb{F}^{\alpha} &\longrightarrow \mathbb{F}^{\beta} \\ y &\longmapsto y \circ f, \end{aligned}$$

and knowing that there is a matrix $\mathcal{G} \in \mathbb{F}^{\beta, (\alpha)}$ such that $\mathbf{Hom}_{\mathbb{F}}(f, \mathbb{F})(y) = \mathcal{G} \cdot y$ for any $y \in \mathbb{F}^{\alpha}$, we get

$$\mathcal{G} \cdot y = y \circ f. \tag{3.2}$$

But, by the above equation, the matrix of the linear mapping $y : \mathbb{F}^{\beta} \longrightarrow \mathbb{F}$ is $\mathcal{H} \in \mathbb{F}^{\beta, 1}$ with $\mathcal{H}_{ji} = y_j$ for any $j \in \beta$. Therefore, we can write the equation (3.2) under the following matrix form

$$\mathcal{G} \cdot \mathcal{H} = \mathcal{F} \cdot \mathcal{H}.$$

Since this is true for any matrix $\mathcal{H} \in \mathbb{F}^{\beta, 1}$, we finally conclude that $\mathcal{G} = \mathcal{F}$.
Let $\alpha, \beta \in \mathbb{N}$ and

$$\begin{aligned} f : \mathbb{F}^{(\beta)} &\longrightarrow \mathbb{F}^{(\alpha)} \\ x &\longmapsto x \cdot \mathcal{F}, \end{aligned}$$

be a linear mapping, where $\mathcal{F} \in \mathbb{F}^{\beta, 1}$. Let

$$\begin{aligned} f' : \mathbb{F}^{\beta} &\longrightarrow \mathbb{F}^{\alpha} \\ y &\longmapsto \mathcal{F}' \cdot y \end{aligned}$$

be another linear mapping. We say that f and f' are **adjoints** if $\mathcal{F}' = \mathcal{F}$. We also say that f (resp. f') is the adjoint of f' (resp. f). The adjoint always exists and is unique because it has the same matrix as the given linear mapping : By proposition 3.1, for any $U, V \in \mathfrak{G}$, the adjoint of an element $f \in \mathbf{Hom}_{\mathfrak{G}}(U, V)$ is the morphism $\mathbf{Hom}_{\mathbb{F}}(f, \mathbb{F})$ of $\mathbf{Hom}_{\mathcal{T}\mathfrak{G}}(\mathbf{Hom}_{\mathbb{F}}(U, \mathbb{F}), \mathbf{Hom}_{\mathbb{F}}(V, \mathbb{F}))$.

THEOREM 3.2. *The functor $\mathbf{Hom}_{\mathbb{F}}(-, \mathbb{F})$ induces a duality between the categories \mathfrak{G} and $\mathcal{T}\mathfrak{G}$.*

Proof. By theorem (2.2), we know that any object of $\mathcal{T}\mathfrak{G}$ is isomorphic to an object of \mathfrak{G} . Therefor, it suffices to show that the functor $\mathbf{Hom}_{\mathbb{F}}(-, \mathbb{F})$ is *faithful* and *full*, i.e., for all $U, V \in \mathfrak{G}$, the mapping

$$\begin{aligned} \mathbf{Hom}_{\mathfrak{G}}(U, V) &\longrightarrow \mathbf{Hom}_{\mathcal{T}\mathfrak{G}}(\mathbf{Hom}_{\mathbb{F}}(V, \mathbb{F}), \mathbf{Hom}_{\mathbb{F}}(U, \mathbb{F})) \\ f &\longmapsto \mathbf{Hom}_{\mathbb{F}}(f, \mathbb{F}) \end{aligned}$$

is injective and surjective (then bijective). We can, without losing generality, take $U = \mathbb{F}^{(\alpha)}$ and $V = \mathbb{F}^{(\beta)}$ where $\alpha, \beta \in \mathbb{N}$.

Surjectivity : we know that an element of $\mathbf{Hom}_{\mathcal{T}\mathfrak{G}}(\mathbf{Hom}_{\mathbb{F}}(V, \mathbb{F}), \mathbf{Hom}_{\mathbb{F}}(U, \mathbb{F}))$ is of the following form

$$\begin{aligned} F : \mathbb{F}^{\alpha} &\longrightarrow \mathbb{F}^{\beta} \\ y &\longmapsto \mathcal{G} \cdot y \end{aligned}$$

where $\mathcal{G} \in \mathbb{F}^{\beta, (\alpha)}$. Then the mapping

$$\begin{aligned} f : \mathbb{F}^{(\beta)} &\longrightarrow \mathbb{F}^{(\alpha)} \\ x &\longmapsto x \cdot \mathcal{G} \end{aligned}$$

is well defined and is an element of $\mathbf{Hom}_{\mathfrak{G}}(U, V)$ which verifies $\mathbf{Hom}_{\mathbb{F}}(f, \mathbb{F}) = F$.

Injectivity : an element $f \in \mathbf{Hom}_{\mathfrak{G}}(U, V)$ is of the form

$$\begin{aligned} f : \mathbb{F}^{(\beta)} &\longrightarrow \mathbb{F}^{(\alpha)} \\ x &\longmapsto x \cdot \mathcal{F} \end{aligned}$$

with $\mathcal{F} = (\mathcal{F}_{ji})_{ji} \in \mathbb{F}^{\beta, (\alpha)}$ and an element $\varphi \in \mathbf{Hom}_{\mathbb{F}}(V, \mathbb{F})$ is of the form

$$\begin{aligned} \varphi : \mathbb{F}^{(\alpha)} &\longrightarrow \mathbb{F} \\ y &\longmapsto \mathcal{G} \cdot y \end{aligned}$$

where $\mathcal{G} = (\mathcal{G}_{ij})_i \in \mathbb{F}^{\alpha, 1}$. Then we have

$$\varphi(f(x)) = \mathcal{G} \cdot (\mathcal{F}(x)) = \mathcal{G} \cdot (x \cdot \mathcal{F}) = \sum_i \sum_j \mathcal{G}_{i1} x_j \mathcal{F}_{ji}.$$

Now, suppose that $\varphi \circ f = 0$ for all $\varphi \in \mathbf{Hom}_{\mathbb{F}}(V, \mathbb{F})$, i.e. $\varphi(f(x)) = 0$ for any $x \in \mathbb{F}^{(\beta)}$. Then

$$\sum_i \sum_j \mathcal{G}_{i1} x_j \mathcal{F}_{ji} = 0 \quad (3.3)$$

for any $x \in \mathbb{F}^{(\beta)}$ and any matrix $\mathcal{G} \in \mathbb{F}^{\alpha,1}$. Given $j \in \beta$ and $i \in \alpha$, set for \mathcal{G} the matrix such that

$$\mathcal{G}_{ij} = \begin{cases} 0 & \text{if } j \neq 1, \\ 1 & \text{if } j = 1. \end{cases}$$

Then (3.3) becomes

$$\mathcal{F}_{ji} = 0$$

for all $(i, j) \in \alpha \times \beta$. Hence $f = 0$. \square

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